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ON ESTIMATION OF A CLASS OF EFFICACY-RELATED PARAMETERS.(U)

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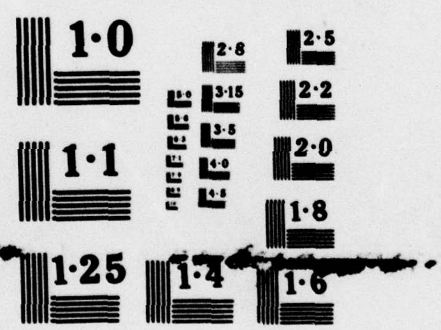
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ON ESTIMATION OF A CLASS OF
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10 ~~by~~ K. F. Cheng ~~and~~ R. J. Serfling

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ABSTRACT

ON ESTIMATION OF A CLASS OF EFFICACY-RELATED PARAMETERS

For specified functions ϕ and ψ and unknown distribution function F with density f, the efficacy-related parameter $T(f) = \int \phi(x)\psi(F(x))f^2(x)dx$ may be estimated by the sample analogue estimator $T(f_n)$ based on an empirical density estimator f_n. For $\{X_i\}$ i.i.d. F and f_n of the form $f_n(x) = n^{-1}\sum_{i=1}^{n} K_n(X_i - x)$, we approximate the estimation error $T(f_n) - T(f)$ by the Gâteaux derivative of the functional $T(\cdot)$ at the "point" f with increment $f_n - f$. In conjunction with stochastic properties of the L_2-norm $||f_n - f||$, this approach leads to characterizations of the stochastic behavior of $T(f_n) - T(f)$. In particular, under mild assumptions on f, we obtain the rate of strong convergence $T(f_n) - T(f) =_{a.s.} O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}})$, which significantly improves previous results in the literature. Also, we establish asymptotic normality with associated Berry-Esséen rates.

Key Phrases: Nonparametric estimation; efficacy; functionals of probability density; strong convergence; asymptotic distribution.

1. Introduction. In nonparametric inference two statistical procedures are ofetn compared by their asymptotic relative efficiency (ARE), which depends on efficacy parameters defined in terms of the underlying probability distribution of the data. An important such efficacy-related functional is

$$T(f) = \int \phi(x)\psi(F(x))f^2(x)dx, \tag{1.1}$$

where f is the underlying probability density function, F is the corresponding cumulative distribution function (cdf), and ϕ and ψ are specified functions. For example, for the case $\phi(x) \equiv \psi(x) \equiv 1$, this functional reduces to $\int f^2(x)dx$, which appears as a factor in the Pitman ARE of various test comparisons involving as one of the tests the Wilcoxon rank sum test, or the Wilcoxon signed rank test, or the Kruskal-Wallis test. Other important special cases of (1.1) are $\int x[2F(x) - 1]f^2(x)dx$, $\int x\{I(-\infty, 0] - I[0, \infty)\}f^2(x)dx$, and $\int (d/dx)\Phi^{-1}(F(x))f^2(x)dx$, where Φ denotes the standard normal cdf. Discussion of these and other examples may be found in Puri and Sen (1971).

Usually little is assumed known about the underlying probability density f, but some enlightenment may be gained by finding the lower bound of the ARE over a specified class of densities. It also becomes of interest to estimate the ARE from the data. In this connection, we explore in this paper the stochastic behavior of certain estimators of the functional T(f) defined by (1.1).

For the special case $\int f^2(x)dx$, a consistent estimate was produced by Lehmann (1963) as a byproduct of an investigation using the signed rank test to construct a confidence interval for the location shift parameter. More generally, Sen (1966) proposed estimators for T(f) in the case that $\phi(x) \equiv 1$ or $\phi(x) = x$ and $\psi(x) = J'(x)$, where J is a score function defining a rank test, and established weak consistency and asymptotic normality of these estimators (under regularity conditions on f and J).

Bhattacharyya and Roussas (1969) suggested estimation of $\int f^2(x)dx$ by $\int f_n^2(x)dx$, where f_n is a kernel-type empirical probability density function for estimation of f based on a sample of size n from f, and established convergence of this estimator in the first and second means. Schuster (1974) investigated strong convergence and established the rate $O(n^{-1/3}\log n)$. He also introduced the alternative estimator, $\int f_n(x)dF_n(x)$, where F_n is the usual empirical cdf, and showed that the two estimators have the same asymptotic almost sure behavior. Ahmad (1976) established asymptotic normality for the latter estimator.

Estimation of the general functional (1.1) has been considered by Ahmad and Lin (1976) and Winter (1978). Winter employs the estimator $T(f_n)$ for T(f), with f_n as above, and establishes strong convergence with rate $O(n^{-1/3}(\log n)\beta_n)$, where $\beta_n \to \infty$, for the case that ϕ is bounded and ψ has a bounded derivative.

In the present treatment, we also consider estimation of T(f) by the sample analogue estimator $T(f_n)$, but we allow greater flexibility in the choice of f_n and we employ a different technique for analysis of $T(f_n)$. Specifically, we approximate $T(f_n) - T(f)$ by the Gâteaux derivative of the functional $T(\cdot)$ at the point f with increment $f_n - f$. By this method we are able to establish significantly improved rates of strong convergence, namely $O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}})$ and under some conditions $O(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}})$, the latter probably optimal. The method also yields asymptotic normality along with associated Berry-Esséen rates. Furthermore, we are able to relax the restrictions on ϕ and ψ imposed by previous authors.

The basic notation, assumptions and method are presented in Section 2. The special case $\phi(x) \equiv \psi(x) \equiv 1$ and f square integrable is treated in Section 3. In Section 4 direct extensions to the following cases are discussed: (a) f has bounded support, ϕ is continuous, ψ has bounded second derivative; (b) f is square integrable, ϕ is bounded, ψ has bounded second derivative. Section 5 treats the general case, dropping all major restrictions on f and ϕ, but at the expense of making the estimator somewhat more complicated. In Section 6 we consider two specific examples of estimators of the simple density functional $\int f^2(x)dx$ and point out certain computational approaches.

2. The basic approach. Let $\{X_i\}$ be independent random variables having density function f. Let f_n be an empirical density function based on $X_1, \ldots, X_n$, and let $\hat{F}_n$ denote the associated cdf obtained by integration of f_n.

We consider estimation of the functional T(f) defined by (1.1) by $T(f_n)$. Following von Mises (1947), let us approximate the estimation error $T(f_n) - T(f)$ by an appropriate Gâteaux derivative. For an arbitrary functional $T(\cdot)$, the Gâteaux derivative of $T(\cdot)$ at the point f with increment g - f, where f and g are "points" in the space of density functions, is defined as

$$T(f; g - f) = \frac{d}{d\lambda}T((1 - \lambda)f + \lambda g)\Big|_{\lambda=0}.$$

For g sufficiently close to f, $T(f; g - f)$ serves as an approximation to $T(g) - T(f)$. In particular, for $T(\cdot)$ given by (1.1) and for $g = f_n$, we find

$$T(f; f_n - f) = 2\int \phi(x)\psi(F(x))f(x)[f_n(x) - f(x)]dx + \int \phi(x)\psi'(F(x))f^2(x)[\hat{F}_n(x) - F(x)]dx, \tag{2.1}$$

assuming that ψ is differentiable.

The usefulness of (2.1) will depend in part on properties of f_n. We shall assume that f_n has the form

$$f_n(x) = n^{-1} \sum_{i=1}^{n} f_{ni}(x), \tag{A1}$$

where the i-th function f_{ni} depends only on the i-th random variable X_i and on n. For example, this structure includes the kernel type f_n in which f_{ni} is of the form $f_{ni}(x) = c_n^{-1}K(c_n^{-1}(x - X_i))$, where K is a specified "kernel" function and $\{c_n\}$ is a sequence of constants tending to 0. Sometimes we shall assume in addition that

$$f_{ni} = f_{ii},\ 1 \le i \le n,\ n = 1, 2, \ldots, \tag{A2}$$

which makes $\{f_n\}$ computable recursively: $f_n = n^{-1}[(n - 1)f_{n-1} + f_{nn}]$. That is, the n-th stage function depends on $X_1, \ldots, X_{n-1}$ only through the result of the (n - 1)-th stage computation.

A key feature of f_n due to (A1) is its structure as an average over the independent elements of the n-th row of a double array of random variables. By (2.1), we readily see that this feature applies as well to the structure of $T(f; f_n - f)$:

(2.2) $$T(f;\, f_n - f) = n^{-1} \sum_{i=1}^{n} T(f;\, f_{ni} - f).$$

Thus we may handle $T(f;f_n - f)$ by routine application of classical probability theory for sums. In the recursive case, that is, when (A2) holds also, there is a further simplication: the problem reduces to averaging over a single sequence of random variables.

The usefulness of (2.1) will depend also upon negligibility of the approximation error $T(f_n) - T(f) - T(f;\, f_n - f)$. In order to show that this quantity is $O_p(n^{-\frac{1}{2}})$, or almost surely (a.s.) $O(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}})$, or the like, we shall use the following "differential inequality." Let $||h||_p$ denote for $0 < p < \infty$ the L_p-norm $(\int|h|^p)^{1/p}$ and for $p = \infty$ the sup-norm $\sup_x|h(x)|$.

LEMMA 2.1. *Let* T *be given by* (1.1). *Assume that either* (a) f *has bounded support and* ϕ *is continuous, or* (b) f *is square integrable and* ϕ *is bounded. Assume that* ψ *has bounded second derivative. Then*

(2.3) $$|T(f_n) - T(f) - T(f;\, f_n - f| \le c_1||f_n - f||_2^2 + c_2||\hat{F}_n - F||_\infty^2,$$

where c_1 *and* c_2 *are constants depending on* f, ϕ *and* ψ *but not on* f_n. *Further, in case* (a) *we may take* $c_2 = 0$.

The proof is routine and omitted. We will exploit the lemma by assuming that f and $\{f_n\}$ are such that the following conditions hold:

(B1) $$n^{\frac{1}{2}}||f_n - f||_2^2 \to_{a.s.} 0,$$

(B2) $$n^{\frac{1}{2}}||\hat{F}_n - F||_\infty^2 \to_{a.s.} 0.$$

Conditions for (B1) have been investigated by Cheng and Serfling (1979) for kernel type f_n. Each of the following is a sufficient requirement on f: (i) f possesses a bounded continuous $L_2(-\infty, \infty)$ derivative; (ii) f is Lipschitz on $(-\infty, \infty)$ and satisfies a tail restriction of form $\int_{|x|>t} f(x)dx = O(t^{-q})$, $t \to \infty$, for some $q > 0$; (iii) the characteristic function of f decreases algebraically of degree $p > 0$, in the sense of Parzen (1962) and Watson and Leadbetter (1963). In each case a suitable choice of kernel K and constants $\{c_n\}$ can be made so as to achieve (B1).

Conditions for (B2) follow from work of Winter (1979), who establishes for suitable f_n the stronger property $n^{\frac{1}{2}}||\hat{F}_n - F||_\infty =_{a.s.} O((\log\log n)^{\frac{1}{2}})$, under the assumption that f possesses a bounded derivative.

It will also be of interest, in connection with Berry-Esséen rates, to have f and f_n satisfy

(C) $$P(n^{\frac{1}{2}}||f_n - f||_2^2 > a_n) = O(a_n),$$

for a sequence of constants a_n tending to 0. The work of Cheng and Serfling noted above also provides (C) under conditions similar to (i), (ii), (iii). However, the analogue of (C) for $||\hat{F}_n - F||_\infty^2$ has not been investigated at this point.

In dealing in Section 5 with the general case of T(f) with f and ϕ unrestricted, our estimator will be a truncated version of $T(f_n)$, namely

$$T_n(f_n) = \int_{-t_n}^{t_n} \phi(x)\psi(\hat{F}_n(x))f_n^2(x)dx,$$

where t_n is a sequence of constants tending to ∞. The corresponding Gâteaux derivative of $T_n(\cdot)$ at f with increment $f_n - f$ is a similiar truncation of $T(f;\, f_n - f)$.

Both $T(f_n)$ and $T_n(f_n)$ are Borel measurable if f_n is a kernel type estimator. In the sequel we assume that $T(f_n)$, $T_n(f_n)$, $T(f;\, f_n - f)$ and $T_n(f;\, f_n - f)$ are Borel measurable without further mention.

The following notation will be needed:

$$\mu_{ni} = E\{T(f;\, f_{ni} - f)\},\ \mu_n = n^{-1}\sum_{i=1}^{n}\mu_{ni};$$

$$\sigma_{ni}^2 = \mathrm{Var}\{T(f;\, f_{ni} - f)\},\ \sigma_n^2 = n^{-1}\sum_{i=1}^{n}\sigma_{ni}^2;$$

$$\gamma_{ni}(\nu) = E|T(f;\, f_{ni} - f) - \mu_{ni}|^{\nu},\ \gamma_n(\nu) = n^{-1}\sum_{i=1}^{n}\gamma_{ni}(\nu).$$

3. <u>The case $\phi \equiv \psi \equiv 1$</u>. In this section the target functional is *simply* $T(f) = \int f^2(x)dx$. *Under the general assumptions* (A), (B), *and* (C) on f and f_n, discussed in Section 2, we characterize the stochastic behavior of $T(f_n)$. Theorem 1 provides the rate of a.s. convergence. Theorem 2 provides asymptotic normality along with an associated Berry-Esséen rate. The hypotheses of the theorems will also entail restrictions directly imposed on the quantities μ_n, σ_n^2, $\gamma_n(\nu)$, etc. These conditions will be further discussed at the conclusion of this section.

THEOREM 1. *Let* f *and* f_n *satisfy* (A1) *and* (B1). *Assume also that*

(3.1) $$\int|f_{ni}(x)|dx \le C,\ \textit{all } i \textit{ and } n,$$

and

(3.2) $$\mu_n = o(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}),\ n \to \infty.$$

Then

(3.3) $$|T(f_n) - T(f)| =_{a.s.} O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}),\ n \to \infty.$$

If, also, (A2) *holds,* $\mu_n = o(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}})$, *and* $\sigma_n^2 \to \sigma^2$, $0 \quad \sigma^2 < \infty$, *then*

$$\overline{\lim_{n\to\infty}} \frac{T(f_n) - T(f)}{(2\sigma^2 n \log\log n)^{\frac{1}{2}}} =_{a.s.} 1. \tag{3.4}$$

PROOF. By Lemma 2.1 and (B1), we have

$$n^{\frac{1}{2}} |T(f_n) - T(f) - T(f;\, f_n - f)| \to_{a.s.} 0. \tag{3.5}$$

In view of (3.2), to complete the proof of (3.3) it suffices to show

$$|T(f;\, f_n - f) - E\{T(f;\, f_n - f)\}| =_{a.s.} O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}). \tag{3.6}$$

By (2.1) and (2.2), represent

$$T(f;\, f_n - f) = n^{-1} \sum_{i=1}^{n} 2\int f(x)[f_{ni}(x) - f(x)]dx. \tag{3.7}$$

By (3.1), the summands in (3.7) are bounded random variables, say bounded by B. Therefore, by Theorem 2 of Hoeffding (1963), we have

$$P(|T(f;\, f_n - f) - E\{T(f;\, f_n - f)\}| \geq t) \leq 2\exp(-2nt^2/B^2),$$

from which (3.6) follows by the Borel-Cantelli lemma.

On the other hand, if f_n satisfies (A2), then $T(f;\, f_n - f)$ may be regarded as the partial sum of independent bounded random variables. Thus (3.4) follows from (3.5), $\mu_n = o(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}})$, and the law of the iterated logarithm of Kolmogorov (1929). □

THEOREM 2. *Let* f *and* f_n *satisfy* (A1) *and* (B1). *Assume also that*

$$\mu_n = o(n^{-\frac{1}{2}}), \tag{3.8a}$$

$$\sigma_n^2 \to \sigma^2,\ 0 < \sigma^2 < \infty, \tag{3.8b}$$

and, for some $\nu > 2$,

(3.8c) $$n\gamma_n(\nu)/(n\sigma_n^2)^{\frac{1}{2}\nu} \to 0.$$

Then

(3.9) $$n^{\frac{1}{2}}[T(f_n) - T(f)] \to_d N(0, \sigma^2).$$

If, also, (C) *holds for a sequence* $\{a_n\}$ *such that* $n^{-\frac{1}{2}} = O(a_n)$, *and* $n\gamma_n(3)/(n\sigma_n^2)^{3/2} = O(n^{-\frac{1}{2}})$, *then (for* Φ *the standard normal cdf)*

(3.10) $$\sup_t |P(n^{\frac{1}{2}}[T(f_n) - T(f)] \le t) - \Phi(t)| = O(a_n).$$

PROOF. We use the following well-known device. For any sequences of random variables $\{\xi_n\}$ and $\{\eta_n\}$ and sequence of positive constants $\{a_n\}$,

$$\sup_t |P(\xi_n \le t) - \Phi(t)| \le \sup_t |P(\eta_n \le t) - \Phi(t)| + O(a_n) + P(|\xi_n - \eta_n| \ge a_n).$$

By this inequality and an argument similar to that for Theorem 1, we reduce the problem to an application of standard central limit theory for double arrays. □

As will be seen below, it suffices for (3.8), and thus for (3.2) also, that f have a bounded second derivative. (Of course, it is understood that f_n must be suitably chosen, also.) If, further, f" is a continuous $L_2(-\infty, \infty)$ function and f_n is of suitable kernel type, then (B1) holds and (C) holds with $a_n = O(n^{-3/10+\epsilon})$, any $\epsilon > 0$. For details on the latter, see Cheng and Serfling (1979).

We now give conditions on f and f_n sufficient for the properties

(3.11) $$\mu_n = o(n^{-\frac{1}{2}}),$$

$$\sigma_n^2 \to \sigma^2, \; 0 < \sigma^2 < \infty, \tag{3.12}$$

$$\gamma_n(3)/n^{\frac{1}{2}}\sigma_n^3 = O(n^{-\frac{1}{2}}). \tag{3.13}$$

We confine attention to f_n of *kernel* type.

LEMMA 3.1. *Let* f *have bounded second derivative. Assume that both* $\int\phi(x)\psi(F(x))f(x)dx$ *and* $\int\phi(x)\psi'(F(x))f^2(x)dx$ *are finite. Let* K *satisfy* $\int zK(z)dz = 0$ *and* $\int z^2|K(z)|dz < \infty$, *and suppose* $c_n = O(n^{-\frac{1}{4}})$. *Then* (3.11) *holds.*

LEMMA 3.2. *Let* f *be bounded and continuous, let* ϕ *be bounded and continuous, and let* ψ *have bounded derivative. Then* σ_n^2 *has finite positive limit and* $\gamma_n(3)$ *is bounded.*

The proofs are routine and may be found, with related results, in Cheng (1979).

4. Some direct extensions. Here we indicate extensions of Section 1 in two directions. For the first case we assume

f has bounded support, say in [a, b];

ϕ is continuous;

ψ has bounded second derivative.

We also assume that the empirical density function f_n has support in [a, b] for large n, which can be arranged by taking f_n to be of kernel type with kernel function having bounded support. Under these assumptions, Theorems 1 and 2 of Section 3 carry over unchanged and by means of similar proofs.

Next we assume, alternatively, that

f is square integrable;

ϕ is bounded;

ψ has bounded second derivative.

In this case both terms in (2.3) are relevant, so that condition (B2) comes into action. With appropriate modifications in this respect, again the assertions of Section 3 carry over.

5. The general case. In this section we simultaneously remove the conditions on ϕ and drop the restrictions on the support of f. We assume only that f is square integrable, and we retain the assumption that ψ has bounded second derivative. *Instead of the estimator* $T(f_n)$, *we employ* the truncated version defined in Section 2, and we introduce the function

$$H(t_n) = \sup_{|x|\le t_n} |\phi(x)|.$$

The differential inequality of Lemma 2.1 now becomes replaced by

$$|T_n(f_n) - T_n(f) - T_n(f;\, f_n - f)| \le cH(t_n)\left[\int_{-t_n}^{t_n} [f_n(x) - f(x)]^2 dx + ||\hat{F}_n - F||_\infty^2\right].$$

Also, the parameters μ_n, σ_n^2, etc. are modified to $\tilde{\mu}_n = E\{T_n(f;\, f_n - f)\}$, $\tilde{\sigma}_n^2 = n\,\mathrm{Var}\{T_n(f;\, f_n - f)\}$, etc.

With modifications along these lines, Theorem 1 of Section 2 carries over to the present situation. Specifically, we add condition (B2) and replace the condition on μ_n by $\tilde{\mu}_n = o(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}H(t_n))$, and assert:

(5.1) $$|T_n(f_n) - T_n(f)| =_{a.s.} O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}H(t_n)).$$

If there exists a choice of t_n such that

(5.2) $$\int_{|x|>t_n} \phi(x)\psi(F(x))f^2(x)dx = O(n^{-\frac{1}{2}}(\log n)H(t_n)),$$

then $T_n(f)$ may be replaced by $T(f)$ in (5.1).

Similarly, by replacing μ_n and σ_n by $\tilde{\mu}_n$ and $\tilde{\sigma}_n$ and adding condition (B2) in Theorem 2, the result carries over with the assertion:

(5.3) $$n^{\frac{1}{2}}[T_n(f_n) - T_n(f)] \to_d N((, \tilde{\sigma}^2).$$

If the left-hand side of (5.2) is $o(n^{-\frac{1}{2}})$, then $T_n(f)$ may be replaced by $T(f)$ in (5.3).

6. Examples and computations. In this section we confine attention to the case $T(f) = \int f^2(x)dx$ and consider f_n to be of kernel type. Two choices of kernel K will be considered.

EXAMPLE 1. *The uniform density as kernel function.* Define $K(x) = \frac{1}{2}$ if $|x| \le 1$, and $K(x) = 0$ otherwise. Then, following an argument of Bhattacharyya and Roussas (1969), $T(f_n)$ may be expressed as a linear combination of order statistics,

$$T(f_n) = (2nc_n)^{-1} + \tfrac{1}{2}(nc_n)^{-2}\sum_{(*)}(2c_n - |X_j - X_i|),$$

where $\sum_{(*)}$ devotes summation over all $1 \le i \le j \le n$ such that $|X_i - X_j| \le 2c_n$. If f has a bounded second derivative which is a continuous $L_2(-\infty, \infty)$ function, and if $c_n = An^{-1/5}$, then by results of Cheng and Serfling (1979) the conditions of Theorems 1 and 2 hold and we have $T(f_n) - T(f) =_{a.s.} O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}})$ as well as $n^{\frac{1}{2}}[T(f_n) - T(f)] \to_d N(0, 4\int f(x)[f(x) - T(f)]^2dx.$ □

EXAMPLE 2. *The triangular function as a kernel function.* Define $K(x) = 1 - x$ if $0 \le x \le 1$, $= 1 + x$ if $-1 \le x \le 0$, $= 0$ otherwise. It can be shown that $T(f_n)$ may be represented as a polynomial function of the differences $|X_i - X_j|$. Also, the same assertions of a.s. convergence and asymptotic normality as in the preceding example apply. □

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Nonparametric estimation; efficacy; functionals of probability density; strong convergence; asymptotic distribution.

20. ABSTRACT

For specified functions ϕ and ψ and unknown distribution function F with density f, the efficacy-related parameter $T(f) = \int\phi(x)\psi(F(x))f^2(x)dx$ may be estimated by the sample analogue estimator $T(f_n)$ based on an empirical density estimator f_n. For $\{X_i\}$ i.i.d. F and f_n of the form $f_n(x) = n^{-1}\sum_{i=1}^n K_n(X_i - x)$, we approximate the estimation error $T(f_n) - T(f)$ by the Gâteaux derivative of the functional $T(\cdot)$ at the "point" f with increment $f_n - f$. In conjunction with stochastic proper[illegible] of the L_2-norm

20. **ABSTRACT (con't)**

$||f_n - f||$, this approach leads to characterizations of the stochastic behavior of $T(f_n) - T(f)$. In particular, under mild assumptions on f, we obtain the rate of strong convergence $T(f_n) - T(f) =_{a.s.} O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}})$, which significantly improves previous results in the literature. Also, we establish asymptotic normality with associated Berry-Esséen rates.